Journal of Engineering Physics and Thermophysics, Vol. 68, No. 1, 1995

DETERMINATION OF THE STRESS-STRAIN STATE OF A COOLING PLANE REGION WITH ACCOUNT FOR A POLYMORPHIC TRANSFORMATION

S. S. Savinskii and R. F. Shakirov

UDC 539.371:669.112

A simplified method for finite-element calculation of the stressed state of a plane region undergoing a polymorphic transformation is described. An exact solution is obtained for a circular region and the exact solution is compared with the numerical one. The effect of an austenite-ferrite transformation on the behavior of the stressed state in the circular region is analyzed.

As a rule, cooling of castings is accompanied by the appearance of inhomogeneous fields induced [1] by inhomogeneity of the temperature field in the casting, nonsimultaneous polymorphic transformations in different parts of the casting, and retardation of shrinkage of the casting by the mold.

In the present work a method is described that is used for simplified calculation of the joint effect of the first two factors on the state of a thin casting of an iron-carbon alloy, cooling in the solid phase. The following simplifications are assumed here:

only the austenite-ferrite transformation is considered;

transformation-induced elastic strains are assumed to be compression-stretching strains;

transformation-induced shear strains are neglected;

the calculations are carried out within unbound linear thermal elasticity;

the mechanical and thermal coefficients are assumed to be constant and independent of temperature.

1. Mathematical Model. The following mathematical model is used: the problem of cooling of the plane region Ω is considered. At the boundary L of the region a constant heat flux is prescribed.

In the region Ω the heat-conduction equation has the following form:

$$T(x, y, t) - a\nabla^2 T(x, y, t) = 0, \quad T(x, y, 0) = T_0, \quad \left.\frac{\partial T}{\partial n}\right|_L = \frac{q}{h}, \tag{1}$$

where $T(x, y, t)$ is the value of the temperature at the point with the coordinate (x, y) at the time t.

The solution of Eq. (1) will be sought by the finite-element method in the projection formulation [2]. For this purpose, the function $T(x, y, t)$ is replaced by an approximate temperature function expressed as the finite series

$$\overline{T}(x, y, t) = \sum_{j=1}^{m} N_j(x, y)\, T_j^*(t), \tag{2}$$

where j is the number of the node; m is the number of nodes in the region Ω; $N_j(x, y)$ are nonzero local functions in a neighborhood of the node j, which later will be referred to as shape functions; $T_j^*(t)$ is the temperature at the node j. To find the node temperatures in Eq. (2), use will be made of the condition of orthogonality of the residue to all shape functions [2]:

$$\int\int_{\Omega} [\overline{T}(x, y, t) - a\nabla^2\overline{T}(x, y, t)]\, N_j(x, y)\, d\Omega = 0, \quad j = 1, 2, \ldots, m. \tag{3}$$

Izhevsk Mechanical Works, Izhevsk, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 68, No. 1, pp. 123-127, January-February, 1995. Original article submitted August 3, 1993.

The substitution of Eq. (2) into the m equations (3) leads to a system of ordinary differential equations relative to the time functions $T_j^*(t)$, which is integrated by the finite-difference method. This results in a system of linear algebraic equations with a band matrix, whose solution is the field of nodal temperatures at discrete moments.

The thermoelastic problem is solved at each time step and two temperature fields are used here, namely, actual and fictitious ones. The fictitious temperature field simulates polymorphic transformations during cooling. The modified functional of the free energy is chosen in the form

$$F = \int \int_{\Omega} \left[\frac{\lambda}{2} \varepsilon_0^2 + \mu \left[u_x^2 + v_y^2 + \frac{1}{2} (u_y + v_x)^2 \right] - \alpha K \varepsilon_0 ((\overline{T} - T_{\text{u}}) + \Delta T^p) \right] d\Omega , \tag{4}$$

where u, v are projections of the displacement vector on the coordinate axes OX and OY; u_x, v_y, u_y, v_x are partial derivatives of the components of the displacement vector with respect to the coordinates; $\varepsilon_0 = u_x + v_y$ is the trace of the strain tensor; ΔT^p is the change in the fictitious temperature; T is the actual temperature.

Assuming that structural changes are accompanied by small local uniform compression or stretching strains (depending on the changes in the density), we can write

$$\nabla T^p = \alpha^{-1} \left(\frac{\Delta V}{V} \right)^p ,$$

where $(\Delta V/V)^p$ is the local change in the volume induced by polymorphic transformations, which must be calculated from the solution of the problem of the local kinetics of the transformations. In what follows, we will restrict ourselves to the very simple concept of the local kinetics of the phases obtained from the equilibrium phase diagram by the rule of segments.

The finite-element discretization of the computation region allows the functional of the free energy (4) to be reduced to the quadratic form [3]. This form is minimized by the formulas of [4]

$$u_j^{k+1} = u_j^k + \beta^k s u_j^k , \quad v_j^{k+1} = v_j^k + \beta^k s v_j^k , \quad j = 1 , 2 , \dots , m , \tag{5}$$

where u_j, v_j are components of the displacement vectors at the j-th node; k is the number of the iteration; su_j^k and sv_j^k are the directions of search; β^k is the length of the step.

The components of the strain and stress tensors are determined for any node of the calculation region using the displacement fields found from Eq. (5):

$$\varepsilon_{xx} = u_x , \quad \varepsilon_{yy} = v_y , \quad \varepsilon_{xy} = \frac{1}{2} (u_y + v_x) ;$$

$$\sigma_{xx} = \lambda \varepsilon_0 + 2\mu \varepsilon_{xx} - K \left[\alpha (T - T_{\text{u}}) + \left(\frac{\Delta V}{V} \right)^p \right] ,$$

$$\sigma_{yy} = \lambda \varepsilon_0 + 2\mu \varepsilon_{yy} - K \left[\alpha (T - T_{\text{u}}) + \left(\frac{\Delta V}{V} \right)^p \right] , \quad \sigma_{xy} - 2\mu \varepsilon_{xy} .$$

The Mises number is used as a criterion of the short-time strength:

$$\sigma_i = \sqrt{(\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2)} < \sigma_y ,$$

where σ_i is the stress intensity; σ_y is the yield point of the material. With this criterion it is possible to predict the regions of possible discontinuities in the material.

2. Analytical Example. An infinite cylinder with a constant heat flux on the surface will be taken as the calculation region. In the present case the temperature problem admits an exact solution as an infinite series of Bessel functions [5]:

$$T(r, t) = \frac{q}{h} R \left[2\,\mathrm{Fo} - \frac{1}{4}\left(1 - \frac{2r^2}{R^2}\right) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0\left(\mu_n \frac{r}{R}\right) \exp(-\mu_n^2\,\mathrm{Fo}) \right], \tag{6}$$

where μ_n are the roots of the characteristic equation $J_1(\mu_n) = 0$.

To solve the elastic problem in the case of a polymorphic transformation, the local change in the volume will be defined as follows:

$$\left(\frac{\Delta V}{V}\right)^p = Q\Phi .$$

The fraction Φ of the new phase is calculated from the linearized diagram of state by the rule of segments:

$$\Phi = A - \frac{B}{T_A - T(r, t)}, \quad A = \frac{c_1}{c_1 - c_2}, \quad B = \frac{c}{c_1 - c_2}(T_A - T_1).$$

By solving the equilibrium equation we obtain an expression for the displacements and the components of the stress tensor in cylindrical coordinates:

$$u(r, t) = \frac{1 + \nu}{3(1 - \nu) r} \left[\alpha \int_0^r T(r, t)\, r dr - B \int_0^r \frac{r dr}{T_A - T(r, t)} \right] +$$

$$+ \frac{(1 + \nu)(1 - 2\nu) r}{3(1 - \nu) R^2} \left[\alpha \int_0^R T(r, t)\, r dr - B \int_0^R \frac{r dr}{T_A - T(r, t)} \right] +$$

$$+ \frac{1 + \nu}{3} r (A - \alpha T_{\mathrm{u}}), \tag{7}$$

$$\sigma_{rr}(r, t) = \frac{E}{3(1 - \nu)} \left[-\frac{\alpha}{r^2} \int_0^r T(r, t)\, r dr + \frac{B}{r^2} \int_0^r \frac{r dr}{T_A - T(r, t)} + \right.$$

$$\left. + \frac{\alpha}{R^2} \int_0^R T(r, t)\, r dr - \frac{B}{R^2} \int_0^R \frac{r dr}{T_A - T(r, t)} \right], \tag{8}$$

$$\sigma_{ff}(r, t) = \frac{E}{3(1 - \nu)} \left[\frac{\alpha}{r^2} \int_0^r T(r, t)\, r dr - \frac{B}{r^2} \int_0^r \frac{r dr}{T_A - T(r, t)} + \right.$$

$$\left. + \frac{\alpha}{R^2} \int_0^R T(r, t)\, r dr - \frac{B}{R^2} \int_0^R \frac{r dr}{T_A - T(r, t)} - \alpha T + \frac{B}{T_A - T(r, t)} \right]. \tag{9}$$

Substitution of temperature function (6) into Eqs. (7)-(9) results in two types of integrals that can be calculated analytically and numerically.

It should be noted that in formulas (8) and (9) for the components of the strain tensor there are terms of ordinary thermal elasticity and terms of polymorphic transformations and under certain conditions the latter can dominate over the former.

3. Comparison of Numerical and Analytical Results. A numerical calculation of the problem of cooling of a circular region containing 289 nodes and 512 simplex finite elements was carried out by the method described in Sec. 1. The initial temperature of the calculation region and the temperature of the unstrained state were assumed to be 842°C. An iron-carbon alloy with the following characteristics was used for the calculations: % C = 0.2%; E

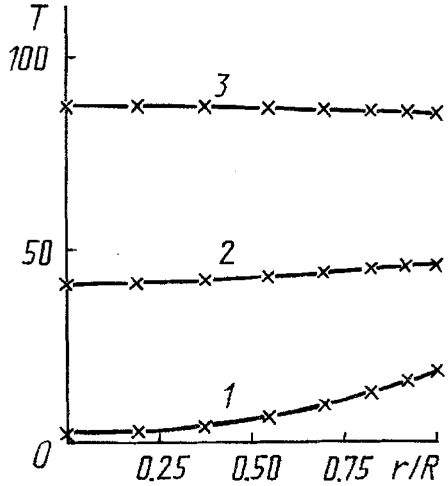

Fig. 1. Plot of the total temperature (the sum of relative, actual, and fictitious temperatures) versus the relative radius: the curves are a numerical calculation. T, °C.

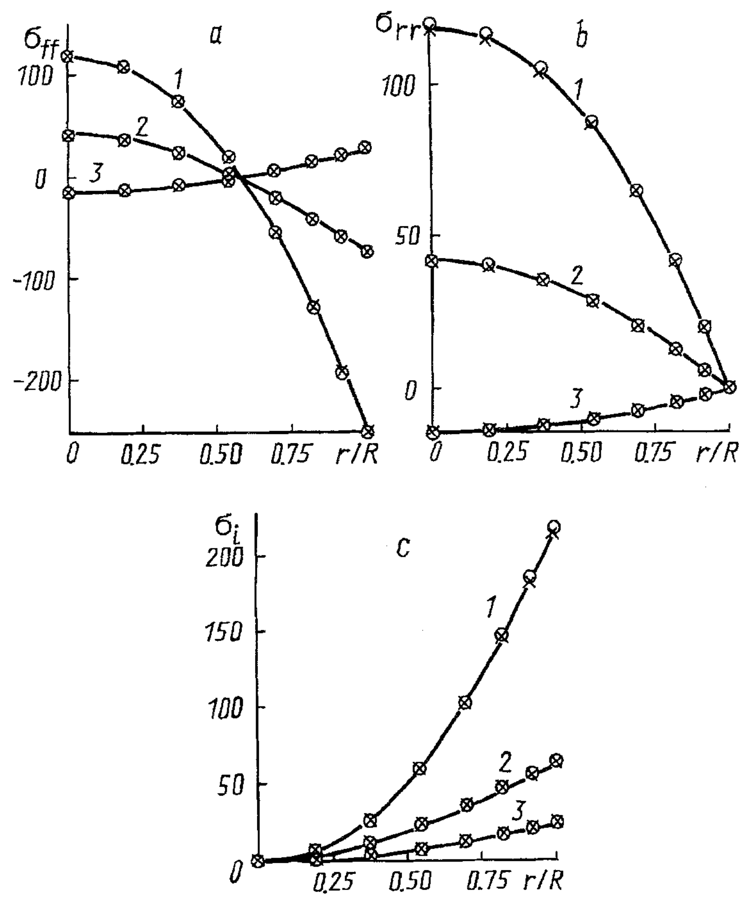

Fig. 2. Plot of tangential (a) and radial (b) components and stress intensity (c) versus the relative radius; ×, O, numerical and exact solutions. σ_{ff}, σ_{rr}, σ_i, MPa.

$= 10^{11}$ N/m^2; $\nu = 0.3$; $G_T = 3 \cdot 10^8$ N/m^2; $\alpha = 3 \cdot 10^{-5}$ 1/deg. For this concentration of the alloy the initial and final temperature of the transformation were 842 and 727°C, respectively.

In cooling, the following type of polymorphic transformations was considered: the austenite-ferrite transformation with the volume effect $Q = 0.0358$. At temperatures lower than the eutectic one pure ferrite was assumed to exist.

In Figs. 1 and 2 one can see results of numerical calculations of the total temperature, components of the stress tensor, and the intensity of stresses with account for polymorphic transformations. Curves 1, 2, and 3 correspond to dimensionless times of 0.110, 0.880, 1.760. By the time Fo = 1.760 austenite has undergone complete transformation.

Inclusion of the transformations in the two-phase austenite-ferrite region (as compared with calculations neglecting the transformations) led to the following results:

the maximum values of the stress intensities were observed in the initial cooling;

the signs of the radial displacements changed and the displacements were directed outside the region;

the signs of the radial and tangential components of the stress tensor changed, the compression regions became stretching regions and vice versa, and absolute values of the components of the stress tensor and the stress intensity have increased severalfold.

Thus, inclusion of the transformations changes markedly the pattern of the stress-strain state of the cooling region. Changes in the sign of radial displacements and radial and tangential components of the strain and stress tensors and a substantial increase in the absolute values of these quantities are both possible.

The closeness of the approximate and exact solutions allows the approximate method to be used in calculations of regions of arbitrary shape.

NOTATION

a, thermal diffusivity; ∇^2, Laplacian; T_0, initial temperature; q, constant heat flux on the boundary; h, thermal conductivity; λ, μ, Lamé parameters; E, elasticity modulus; ν, Poisson coefficient; K, volume compression coefficient; α, volume heat expansion coefficient; T_u, temperature of the unstrained state; ΔT^p, change in the fictitious temperature; u, v, components of the displacement vectors; G_i, intensity of stresses; G_y, yield point of the material; $\mathrm{Fo} = at/R^2$, Fourier number; R, radius of the cylinder; Q, volume effect of the transformation; Φ, fraction of the new phase; c, concentration of carbon in the alloy; T_A, temperature of the $\alpha-\gamma$ transition in pure iron; c_2, maximum solubility of carbon in ferrite at the eutectic temperature: c_1, concentration of carbon in the eutectoid; T_1, temperature of the eutectoid transformation.

REFERENCES

1. A. V. Lakedemonskii, et al., Casting Defects and Methods for Their Elimination [in Russian], Moscow (1972).
2. O. Zenkevich and C. Morgan, Finite Elements and Approximations [Russian translation], Moscow (1984).
3. S. S. Savinskii, R. F. Shakirov, V. V. Okat'ev, and V. P. Votyakov, Problemy Prochnosti, No. 10, 79-81 (1991).
4. G. Recleitis et al., Optimization in Technology [Russian translation], Moscow (1986).
5. A. V. Luikov, Heat Conduction Theory [in Russian], Moscow (1967).